Ergodic Theory - Week 4

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Birkhoff's pointwise ergodic theorem

P1. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, $f \in L^1(\mu)$ and $a \in \mathbb{R}$. Show that, for almost all $x \in X$, the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(na) f(T^n x)$$

exists.

P2. In this exercise, we study the ergodic theorem for non-integrable functions.

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system. Suppose $f \geq 0$ is a measurable function such that $\int f \, d\mu = +\infty$ and define

$$f^*(x) = \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

as well as the set

$$A = \{x \in X \colon f^*(x) < +\infty\}$$

(a) Show that f^* is T-invariant.
Hint: Use the identity

$$\frac{N+1}{N} \left(\frac{1}{N+1} \sum_{n=0}^{N} f(T^n x) \right) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n (Tx)) + \frac{1}{N} f(x).$$

(b) Show that the function $f^* \cdot \mathbb{1}_A$ is constant almost everywhere on X.

(c) Conclude that $\mu(A) = 0$ and thus

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty$$

for almost all $x \in X$.
Hint: Construct an increasing sequence of bounded functions f_m that converges to f pointwise.

P3. Consider the system $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu, T)$ where μ is the Lebesgue measure and $T(x) = qx \bmod 1$ for $q \geq 2$ a fixed integer. If $x \in \mathbb{R}$ we say that x is written in its q-ary digit expansion if

$$x = \lfloor x \rfloor + \sum_{j=1}^{\infty} \frac{a_j(x)}{q^j},$$

for $\{a_j\}_{j \in \mathbb{N}} \subseteq \{0, 1, \ldots, q-1\}$, so that for all $J \in \mathbb{N}$ there exists $j \geq J$ such that[1] $a_j \neq q - 1$.

[1] In base 10, for example, the numbers 0.349999999999999... and 0.35 are equal and we make the convention that we only keep the second representation for this number.

(a) For $x \in \mathbb{R}$, prove that $a_k(x) = i$ if and only if $T^{k-1}x \in [\frac{i}{q}, \frac{i+1}{q})$ with $i \in \{0, 1, \ldots, q-1\}$. Additionally, prove that $a_k(Tx) = a_{k+1}(x)$ for all $x \in \mathbb{R}$ and $k \geq 1$.

(b) Let $c_1, \ldots, c_k$ be a collection of digits in $\{0, \ldots, q-1\}$. Show that there exists a unique $i \in \{0, \ldots, q^k - 1\}$, such that

$$\{a_1(x) = c_1, \ldots, a_k(x) = c_k\} \text{ if and only if } \{x\} \in \left[\frac{i}{q^k}, \frac{i+1}{q^k}\right).$$

Derive the equivalence

$$\{a_{n+1}(x) = c_1, \ldots, a_{n+k}(x) = c_k\} \text{ if and only if } \{T^n x\} \in \left[\frac{i}{q^k}, \frac{i+1}{q^k}\right).$$

(c) We say that a number x is normal in base q if for any finite pattern of digits $\{c_1, \ldots, c_k\} \in \{0, \ldots, q-1\}^k$, we have

$$\lim_{N \to +\infty} \frac{|\{n \leq N \colon a_n(x) = c_1, \ldots, a_{n+k-1}(x) = c_k\}|}{N} = \frac{1}{q^k},$$

where $a_n(x)$ are the digits of x in its base q expansion. Namely, all patterns with k digits appear with the same frequency. Show that x is q-normal if and only if the sequence $\{q^n x\}$ is uniformly distributed mod 1.

Hint: To prove uniform distribution, verify the definition first for intervals of the form $[i/q^k, (i+1)/q^k)$ and then approximate a general interval by intervals with endpoints rational numbers, whose denominators are powers of q.